Remarks on curve classes on rationally connected varieties

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To Joe Harris, on his 60th birthday

1 Introduction

Let X be a smooth complex projective variety. Define

$$Z^{2i}(X) = \frac{\mathrm{Hdg}^{2i}(X, \mathbb{Z})}{H^{2i}(X, \mathbb{Z})_{alg}}, \tag{1}$$

where $\mathrm{Hdg}^{2i}(X, \mathbb{Z})$ is the space of integral Hodge classes on X and $H^{2i}(X, \mathbb{Z})_{alg}$ is the subgroup of $H^{2i}(X, \mathbb{Z})$ generated by classes of codimension i closed algebraic subsets of X.

These groups measure the defect of the Hodge conjecture for integral Hodge classes, hence they are trivial for $i = 0$, 1 and $n = \dim X$, but in general they can be nonzero by [1]. Furthermore they are torsion if the Hodge conjecture for *rational* Hodge classes on X of degree $2i$ holds. In addition to the previously mentioned case, this happens when $i = n-1$, $n = \dim X$, due to the Lefschetz theorem on $(1,1)$-classes and the hard Lefschetz isomorphism (cf. [23]). We will call classes in $\mathrm{Hdg}^{2n-2}(X, \mathbb{Z})$ "curve classes", as they are also degree 2 homology classes.

Note that the Kollár counterexamples (cf. [14]) to the integral Hodge conjecture already exist for curve classes (that is degree 4 cohomology classes in this case) on projective threefolds, unlike the Atiyah-Hirzebruch examples which work for degree 4 integral Hodge classes in higher dimension.

It is remarked in [21], [23] that the two groups

$$Z^4(X),\ Z^{2n-2}(X),\ n := \dim X$$

are birational invariants. (For threefolds, this is the same group, but not in higher dimension.) The nontriviality of these birational invariants for rationally connected varieties is asked in [23]. Still more interesting is the nontriviality of these invariants for unirational varieties, having in mind the Lüroth problem (cf. [3], [2], [4]).

Concerning the group $Z^4(X)$, Colliot-Thélène and the author proved in [8], building on the work of Colliot-Thélène and Ojanguren [5], that it can be nonzero for unirational varieties starting from dimension 6. What happens in dimensions 5 and 4 is unknown (the four dimensional case being particularly challenging in our mind), but in dimension 3, there is the following result proved in [22]:

Theorem 1.1. *(Voisin 2006) Let X be a smooth projective threefold which is either uniruled or Calabi-Yau. Then the group $Z^4(X)$ is equal to* 0.

This result, and in particular the Calabi-Yau case, implies that the group $Z^6(X)$ is also 0 for a Fano fourfold X which admits a smooth anticanonical divisor. Indeed, a smooth anticanonical divisor $j : Y \hookrightarrow X$ is a Calabi-Yau threefold, so that we have $Z^4(Y) = 0$ by Theorem 1.1 above. As $H^2(Y, \mathcal{O}_Y)$, every class in $H^4(Y, \mathbb{Z})$ is a Hodge class, and it follows

that $H^4(Y,\mathbb{Z}) = H^4(Y,\mathbb{Z})_{alg}$. As the Gysin map $j_* : H^4(Y,\mathbb{Z}) \to H^6(X,\mathbb{Z})$ is surjective by the Lefschetz theorem on hyperplane sections, it follows that $H^6(X,\mathbb{Z}) = H^6(X,\mathbb{Z})_{alg}$, and thus $Z^6(X) = 0$.

In the paper [11], it was proved more generally that if X is any Fano fourfold, the group $Z^6(X)$ is trivial. Similarly, if X is a Fano fivefold on index 2, the group $Z^8(X)$ is trivial.

These results have been generalized to higher dimensional Fano manifolds of index $n-3$ and dimension ≥ 8 by Enrica Floris [9] who proves the following result:

Theorem 1.2. *Let X be a Fano manifold over $\mathbb{C}$ of dimension $n \geq 8$ and index $n-3$. then the group $Z^{2n-2}(X)$ is equal to 0: Equivalently, any integral cohomology class of degree $2n-2$ on X is algebraic.*

The purpose of this note is to provide a number of evidences for the vanishing of the group $Z^{2n-2}(X)$, for any rationally connected variety over $\mathbb{C}$. Note that in this case, since $H^2(X,\mathcal{O}_X) = 0$, the Hodge structure on $H^2(X,\mathbb{Q})$ is trivial, and so is the Hodge structure on $H^{2n-2}(X,\mathbb{Q})$, so that $Z^{2n-2}(X) = H^{2n-2}(X,\mathbb{Z})/H^{2n-2}(X,\mathbb{Z})_{alg}$. We will first prove the following two results.

Proposition 1.3. *The group $Z^{2n-2}(X)$ is locally deformation invariant for rationally connected manifolds X.*

Let us explain the meaning of the statement. Consider a smooth projective morphism $\pi : \mathcal{X} \to B$ between connected quasi-projective complex varieties, with n dimensional fibers. Recall from [15] that if one fiber $X_b := \pi^{-1}(b)$ is rationally connected, so is every fiber. Let us endow everything with the usual topology. Then the sheaf $R^{2n-2}\pi_*\mathbb{Z}$ is locally constant on B. On any Euclidean open set $U \subset B$ where this local system is trivial, the group $Z^{2n-2}(X_b)$, $b \in U$ is the finite quotient of the *constant* group $H^{2n-2}(X_b,\mathbb{Z})$ by its subgroup $H^{2n-2}(X_b,\mathbb{Z})_{alg}$. To say that $Z^{2n-2}(X_b)$ is locally constant means that on open sets U as above, the subgroup $H^{2n-2}(X_b,\mathbb{Z})_{alg}$ of the constant group $H^{2n-2}(X_b,\mathbb{Z})$ does not depend on b.

It follows from the above result that the vanishing of the group $Z^{2n-2}(X)$ for X a rationally connected manifold reduces to the similar statement for X defined over a number field.

Let us now define an l-adic analogue $Z^{2n-2}(X)_l$ of the group $Z^{2n-2}(X)$ (cf. [6], [7]). Let X be a smooth projective variety defined over a field K which in the sequel will be either a finite field or a number field. Let $\overline{K}$ be an algebraic closure of K. Any cycle $Z \in CH^s(X_{\overline{K}})$ is defined over a finite extension of K. Let l be a prime integer different from $p = \mathrm{char}\, K$ if K is finite. It follows that the cycle class

$$cl(Z) \in H^{2s}_{et}(X_{\overline{K}}, \mathbb{Q}_l(s))$$

is invariant under a finite index subgroup of $\mathrm{Gal}(\overline{K}/K)$.

Classes satisfying this property are called Tate classes. The Tate conjecture for finite fields asserts the following:

Conjecture 1.4. *(cf. [18] for a recent account) Let X be smooth and projective over a finite field K. The cycle class map gives for any s a surjection*

$$cl : CH^{2s}(X_{\overline{K}}) \otimes \mathbb{Q}_l \to H^{2s}(X_{\overline{K}}, \mathbb{Q}_l(s))_{Tate}.$$

Note that the cycle class defined on $CH^s(X_{\overline{K}})$ takes in fact values in $H^{2s}(X_{\overline{K}}, \mathbb{Z}_l(s))$, and more precisely in the subgroup $H^{2s}(X_{\overline{K}}, \mathbb{Z}_l(s))_{Tate}$ of classes invariant under a finite index subgroup of $\mathrm{Gal}(\overline{K}/K)$. We thus get for each i a morphism

$$cl^i : CH^{2i}(X_{\overline{K}}) \otimes \mathbb{Z}_l \to H^{2i}(X_{\overline{K}}, \mathbb{Z}_l(i))_{Tate}.$$

We can thus introduce the following variant of the groups $Z^{2i}(X)$:

$$Z^{2i}_{et}(X)_l := H^{2i}_{et}(X_{\overline{K}}, \mathbb{Z}_l(i))_{Tate}/\mathrm{Im}\, cl^i.$$

An argument similar to the one used for the proof of Proposition 1.3 will lead to the following result:

Proposition 1.5. *Let X be a smooth rationally connected variety defined over a number field K, with ring of integers $\mathcal{O}_K$. Assume given a projective model $\mathcal{X}$ of X over $\operatorname{Spec}\mathcal{O}_K$. Fix a prime integer l. Then except for finitely many $p \in \operatorname{Spec}\mathcal{O}_K$, the group $Z^{2n-2}_{et}(X)_l$ is isomorphic to the group $Z^{2n-2}_{et}(X_p)_l$.*

In the course of the paper, we will also consider variants $Z^{2n-2}_{rat}(X)$, resp. $Z^{2n-2}_{et,rat}(X)_l$ of the groups $Z^{2n-2}(X)$, resp. $Z^{2n-2}_{et}(X)_l$, obtained by taking the quotient of the group of integral Hodge classes (resp. integral l-adic Tate classes) by the subgroup generated by classes of *rational* curves. This variant is suggested by Kollár's paper (cf. [16, Question 3, (1)]). By the same arguments, these groups are also deformation and specialization invariants for rationally connected varieties.

Our last result is conditional but it strongly suggests the vanishing of the group $Z^{2n-2}(X)$ for X a smooth rationally connected variety over $\mathbb{C}$. Indeed, we will prove using the main result of [19] and the two propositions above the following consequence of Theorem 1.5:

Theorem 1.6. *Assume Tate's conjecture 1.4 holds for degree 2 Tate classes on smooth projective surfaces defined over a finite field. Then the group $Z^{2n-2}(X)$ is trivial for any smooth rationally connected variety X over $\mathbb{C}$.*

Thanks. *I thank the organizers of the beautiful conference "A celebration of algebraic geometry" for inviting me there. I also thank Jean-Louis Colliot-Thélène, Olivier Debarre and János Kollár for useful discussions.*

It is a pleasure to dedicate this note to Joe Harris, whose influence on the subject of rational curves on algebraic varieties (among other topics!) is invaluable.

2 Deformation and specialization invariance

Proof of Proposition 1.3. We first observe that, due to the fact that relative Hilbert schemes parameterizing curves in the fibers of B are a countable union of varieties which are projective over B, given a simply connected open set $U \subset B$ (in the classical topology of B), and a class $\alpha \in \Gamma(U, R^{2n-2}\pi_*\mathbb{Z})$ such that α_t is algebraic for $t \in V$, where V is a smaller nonempty open set $V \subset U$, then α_t is algebraic for any $t \in U$.

To prove the deformation invariance, we just need using the above observation to prove the following:

Lemma 2.1. *Let $t \in U \subset B$, and let $C \subset X_t$ be a curve and let $[C] \in H^{2n-2}(X_t, \mathbb{Z}) \cong \Gamma(U, R^{2n-2}\pi_*\mathbb{Z})$ be its cohomology class. Then the class $[C]_s$ is algebraic for s in a neighborhood of t in U.*

Proof of Lemma 2.1. By results of [15], there are rational curves $R_i \subset X_t$ with ample normal bundle which meet C transversally at distinct points, and with arbitrary tangent directions at these points. We can choose an arbitrarily large number D of such curves with generically chosen tangent directions at the attachment points. We then know by [10, §2.1] that the curve $C' = C \cup_{i \leq D} R_i$ is smoothable in X_t to a smooth unobstructed curve $C'' \subset X_t$, that is $H^1(C'', N_{C''/X_t}) = 0$. This curve C'' then deforms with X_t (cf. [12], [13, II.1]) in the sense that the morphism from the deformation of the pair (C'', X_t) to B is smooth, and in particular open. So there is a neighborhood of V of t in U such that for $s \in V$, there is a curve $C''_s \subset X_s$ which is a deformation of $C'' \subset X_t$. The class $[C''_s] = [C'']_s$ is thus algebraic on X_s. On the other hand, we have

$$[C''] = [C'] = [C] + \sum_i [R_i].$$

As the R_i's are rational curves with positive normal bundle, they are also unobstructed, so that the classes $[R_i]_s$ also are algebraic on X_s for s in a neighborhood of t in U. Thus

$[C]_s = [C'']_s - \sum_i [R_i]_s$ is algebraic on X_s for s in a neighborhood of t in U. The lemma, hence also the proposition, is proved.

■

Remark 2.2. There is an interesting variant of the group $Z^{2n-2}(X)$, which is suggested by Kollár (cf. [16]) given by the following groups:

$$Z^{2n-2}_{rat}(X) := H^{2n-2}(X,\mathbb{Z})/\langle [C],\, C \text{ rational curve in } X\rangle.$$

Here, by a rational curve, we mean an irreducible curve whose normalization is rational. These groups are of torsion for X rationally connected, as proved by Kollár ([13, Theorem 3.13 p 206]). It is quite easy to prove that they are birationally invariant.

The proof of Proposition 1.3 gives as well the following result (already noticed by Kollár [16]) :

Variant 2.3. *If $\mathcal{X} \to B$ is a smooth projective morphism with rationally connected fibers, the groups $Z^{2n-2}_{rat}(\mathcal{X}_t)$ are local deformation invariants.*

Let us give one application of Proposition 1.3 (or rather its proof) and/or its variant 2.3. Let X be a smooth projective variety of dimension $n+r$, with $n \geq 3$ and let $\mathcal{E}$ be an ample vector bundle of rank r on X. Let $C_1, \ldots, C_k$ be smooth curves in X whose cohomology classes generate the group $H^{2n+2r-2}(X,\mathbb{Z})$. For $\sigma \in H^0(X,\mathcal{E})$, we denote by X_σ the zero locus of σ. When $\mathcal{E}$ is generated by sections, X_σ is smooth of dimension n for general σ.

Theorem 2.4. *1) Assume that the sheaves $\mathcal{E} \otimes \mathcal{I}_{C_i}$ are generated by global sections for $i = 1, \ldots, k$. Then if X_σ is smooth rationally connected for general σ, the group $Z^{2n-2}(X_\sigma)$ vanishes for any σ such that X_σ is smooth of dimension n.*

2) Under the same assumptions as in 1), assume the curves $C_i \subset X$ are rational. Then if X_σ is smooth rationally connected for general σ, the group $Z^{2n-2}_{rat}(X_\sigma)$ vanishes for any σ such that X_σ is smooth of dimension n.

Proof. 1) Let $j_\sigma : X_\sigma \to X$ be the inclusion map. Since $n \geq 3$ and $\mathcal{E}$ is ample, by Sommese's theorem [20], the Gysin map $j_{\sigma *} : H^{2n-2}(X_\sigma,\mathbb{Z}) \to H^{2n+2r-2}(X,\mathbb{Z})$ is an isomorphism. It follows that the group $H^{2n-2}(X_\sigma,\mathbb{Z})$ is a constant group. In order to show that $Z^{2n-2}(X_\sigma)$ is trivial, it suffices to show that the classes $(j_{\sigma *})^{-1}([C_i])$ are algebraic on X_σ since they generate $H^{2n-2}(X_\sigma,\mathbb{Z})$. Since the X_σ's are rationally connected, Theorem 1.3 tells us that it suffices to show that for each i, there exists a $\sigma(i)$ such that $X_{\sigma(i)}$ is smooth n-dimensional and that the class $(j_{\sigma(i)*})^{-1}([C_i])$ is algebraic on $X_{\sigma(i)}$.

It clearly suffices to exhibit one smooth $X_{\sigma(i)}$ containing C_i, which follows from the following lemma:

Lemma 2.5. *Let X be a variety of dimension $n+r$ with $n \geq 2$, $C \subset X$ be a smooth curve, $\mathcal{E}$ be a rank r vector bundle on X such that $\mathcal{E} \otimes \mathcal{I}_C$ is generated by global section. Then for a generic $\sigma \in H^0(X, \mathcal{E} \otimes \mathcal{I}_C)$, the zero set X_σ is smooth of dimension n.*

Proof. The fact that X_σ is smooth of dimension n away from C is standard and follows from the fact that the incidence set $(\sigma, x) \in \mathbb{P}(H^0(X, \mathcal{E} \otimes \mathcal{I}_C)) \times (X \setminus C),\, \sigma(x) = 0\}$ is smooth of dimension $n+N$, where $N := \dim \mathbb{P}(H^0(X, \mathcal{E} \otimes \mathcal{I}_C))$. It thus suffices to check the smoothness along C for generic σ.

This is checked by observing that since $\mathcal{E} \otimes \mathcal{I}_C$ is generated by global sections, its restriction $\mathcal{E} \otimes N^*_{C/X}$ is also generated by global sections. This implies that for each point $c \in C$, the condition that X_σ is singular at c defines a codimension n closed algebraic subset P_c of $P := \mathbb{P}(H^0(X, \mathcal{E} \otimes \mathcal{I}_C))$, determined by the condition that $d\sigma_c : N_{C/X,c} \to \mathcal{E}_c$ is not surjective. Since $\dim C = 1$, the union of the P_c's cannot be equal to P if $n \geq 2$. ■

This concludes the proof of 1) and the proof of 2) works exactly in the same way. ■

Let us finish this section with the proof of Proposition 1.5.

Proof of Proposition 1.5. Let $p \in \operatorname{Spec} \mathcal{O}_K$, with residue field $k(p)$. Assume $\mathcal{X}_p$ is smooth. For l prime to $\operatorname{char} k(p)$, the (adequately constructed) specialization map

$$H^{2n-2}_{et}(X_{\overline{K}}, \mathbb{Z}_l(n-1)) \to H^{2n-2}_{et}(\mathcal{X}_{\overline{p}}, \mathbb{Z}_l(n-1)) \tag{2}$$

is then an isomorphism (cf. [17, Chapter VI, §4]).

Observe also that since $X_{\overline{K}}$ is rationally connected, the rational étale cohomology group $H^{2n-2}_{et}(X_{\overline{K}}, \mathbb{Q}_l(n-1))$ is generated over $\mathbb{Q}_l$ by curve classes. Hence the same is true for $H^{2n-2}_{et}(\mathcal{X}_{\overline{p}}, \mathbb{Q}_l(n-1))$. Thus the whole cohomology groups

$$H^{2n-2}_{et}(X_{\overline{K}}, \mathbb{Z}_l(n-1)),\ H^{2n-2}_{et}(\mathcal{X}_{\overline{p}}, \mathbb{Z}_l(n-1))$$

consist of Tate classes, and (2) gives an isomorphism

$$H^{2n-2}_{et}(X_{\overline{K}}, \mathbb{Z}_l(n-1))_{Tate} \to H^{2n-2}_{et}(\mathcal{X}_{\overline{p}}, \mathbb{Z}_l(n-1))_{Tate}. \tag{3}$$

In order to prove Proposition 1.5, it thus suffices to prove the following:

Lemma 2.6. *1) For almost every $p \in \operatorname{Spec} \mathcal{O}_K$, the fiber $\mathcal{X}_{\overline{p}}$ is smooth and separably rationally connected.*

2) If $\mathcal{X}_{\overline{p}}$ is smooth and separably rationally connected, for any curve $C_{\overline{p}} \subset \mathcal{X}_{\overline{p}}$, the inverse image $[C_{\overline{p}}]_{\overline{K}} \in H^{2n-2}_{et}(X_{\overline{K}}, \mathbb{Z}_l(n-1))$ of the class $[C_{\overline{p}}] \in H^{2n-2}_{et}(\mathcal{X}_{\overline{p}}, \mathbb{Z}_l(n-1))$ via the isomorphism (3) is the class of a 1-cycle on $X_{\overline{K}}$.

Proof. 1) When the fiber $\mathcal{X}_p$ is smooth, the separable rational connectedness of $\mathcal{X}_{\overline{p}}$ is equivalent to the existence of a smooth rational curve $C_{\overline{p}} \cong \mathbb{P}^1_{\overline{k(p)}}$ together with a morphism $\phi : C_{\overline{p}} \to \mathcal{X}_{\overline{p}}$ such that the vector bundle $\phi^* T_{\mathcal{X}_{\overline{p}}}$ on $\mathbb{P}^1_{\overline{k(p)}}$ is a direct sum $\oplus_i \mathcal{O}_{\mathbb{P}^1_{\overline{k(p)}}}(a_i)$ where all a_i are positive. Equivalently

$$H^1(\mathbb{P}^1_{\overline{k(p)}}, \phi^* T_{\mathcal{X}_{\overline{p}}}(-2)) = 0. \tag{4}$$

The smooth projective variety $X_{\overline{K}}$ being rationally connected in characteristic 0, it is separably rationally connected, hence there exists a finite extension K' of K, a curve C and a morphism $\phi : C \to X$ defined over K', such that $C \cong \mathbb{P}^1_{K'}$ and $H^1(\mathbb{P}^1_{K'}, \phi^* T_{X_{K'}}(-2)) = 0$.

We choose a model

$$\Phi : \mathcal{C} \cong \mathbb{P}^1_{\mathcal{O}_{K'}} \to \mathcal{X}'$$

of C and ϕ defined over a Zariski open set of $\operatorname{Spec} \mathcal{O}_{K'}$. By upper-semi-continuity of cohomology, the vanishing (4) remains true after restriction to almost every closed point $p \in \operatorname{Spec} \mathcal{O}_{K'}$, which proves 1).

2) The proof is identical to the proof of Proposition 1.3: we just have to show that the curve $C_{\overline{p}} \subset \mathcal{X}_{\overline{p}}$ is algebraically equivalent in $\mathcal{X}_{\overline{p}}$ to a difference $C''_{\overline{p}} - \sum_i R_{i,\overline{p}}$, where each curve $C'''_{\overline{p}}$, resp. $R_{i,\overline{p}}$ (they are in fact defined over a finite extension $k(p)'$ of $k(p)$), lifts to a curve C''', resp. R_i in $X_{K'}$ for some finite extension K' of K.

Assuming the curves $C'''_{\overline{p}}$, $R_{i,\overline{p}}$ are smooth, the existence of such a lifting is granted by the condition $H^1(C'''_{\overline{p}}, N_{C'''_{\overline{p}}/\mathcal{X}_{\overline{p}}}) = 0$, resp. $H^1(R_{i,\overline{p}}, N_{R_{i,\overline{p}}/\mathcal{X}_{\overline{p}}}) = 0$.

Starting from $C \subset \mathcal{X}_{\overline{p}}$ where $\mathcal{X}_{\overline{p}}$ is separably rationally connected over $\overline{p}$, we obtain such curves $C'''_{\overline{p}}$, $R_{i,\overline{p}}$ as in the previous proof, applying [10, §2.1]. ∎

The proof of Proposition 1.5 is finished. ∎

Again, this proof leads as well to the proof of the specialization invariance of the l-adic analogues $Z^{2n-2}_{et,rat}(X)_l$ of the groups $Z^{2n-2}_{rat}(X)$ introduced in Remark 2.2.

Variant 2.7. *Let X be a smooth rationally connected variety defined over a number field K, with ring of integers $\mathcal{O}_K$. Assume given a projective model $\mathcal{X}$ of X over $\operatorname{Spec} \mathcal{O}_K$. Fix a prime integer l. Then for any $p \in \operatorname{Spec} \mathcal{O}_K$ such that $\mathcal{X}_{\overline{p}}$ is smooth separably connected, the group $Z^{2n-2}_{et,rat}(X)_l$ is isomorphic to the group $Z^{2n-2}_{et,rat}(X_p)_l$.*

3 Consequence of a result of Chad Schoen

In [19], Chad Schoen proves the following theorem:

Theorem 3.1. *Let X be a smooth projective variety of dimension n defined over a finite field k of characteristic p. Assume that the Tate conjecture holds for degree 2 Tate classes on smooth projective surfaces defined over a finite extension of k. Then the étale cycle class map:*

$$cl : CH^{n-1}(X_{\overline{k}}) \otimes \mathbb{Z}_l \to H^{2n-2}(X_{\overline{k}}, \mathbb{Z}_l(n-1))_{Tate}$$

is surjective, that is $Z^{2n-2}_{et}(X)_l = 0$.

In other words, the Tate conjecture 1.4 for degree 2 *rational* Tate classes implies that the groups $Z^{2n-2}_{et}(X)_l$ should be trivial for all smooth projective varieties defined over finite fields. This is of course very different from the situation over $\mathbb{C}$ where the groups $Z^{2n-2}(X)$ are known to be possibly nonzero.

Remark 3.2. There is a similarity between the proof of Theorem 3.1 and the proof of Theorem 1.1. Schoen proves that given an integral Tate class α on X (defined over a finite field), there exist a smooth complete intersection surface $S \subset X$ and an integral Tate class β on S such that $j_{S*}\beta = \alpha$ where j_S is the inclusion of S in X. The result then follows from the fact that if the Tate conjecture holds for degree 2 rational Tate classes on S, it holds for degree 2 integral Tate classes on S.

I prove that for X a uniruled or Calabi-Yau, and for $\beta \in Hdg^4(X, \mathbb{Z})$ there exists surfaces $S_i \overset{j_{S_i}}{\hookrightarrow} X$ (in an adequately chosen linear system on X) and integral Hodge classes $\beta_i \in Hdg^2(S_i, \mathbb{Z})$ such that $\alpha = \sum_i j_{S_i*}\beta$. The result then follows from the Lefschetz theorem on $(1,1)$-classes applied to the β_i.

We refer to [7] for some comments on and other applications of Schoen's theorem, and conclude this note with the proof of the following theorem (cf. Theorem 1.6 of the introduction).

Theorem 3.3. *Assume Tate's conjecture 1.4 holds for degree 2 Tate classes on smooth projective surfaces defined over a finite field. Then the group $Z^{2n-2}(X)$ is trivial for any smooth rationally connected variety X over $\mathbb{C}$.*

Proof. We first recall that for a smooth rationally connected variety X, the group $Z^{2n-2}(X)$ is equal to the quotient $H^{2n-2}(X, \mathbb{Z})/H^{2n-2}(X, \mathbb{Z})_{alg}$, due to the fact that the Hodge structure on $H^{2n-2}(X, \mathbb{Q})$ is trivial. In fact, we have more precisely

$$H^{2n-2}(X, \mathbb{Q}) = H^{2n-2}(X, \mathbb{Q})_{alg}$$

by hard Lefschetz theorem and the fact that

$$H^2(X, \mathbb{Z}) = H^2(X, \mathbb{Z})_{alg}$$

by the Lefschetz theorem on $(1,1)$-classes.

Next, in order to prove that $Z^{2n-2}(X)$ is trivial, it suffices to prove that for each l, the group $Z^{2n-2}(X) \otimes \mathbb{Z}_l = H^{2n-2}(X, \mathbb{Z}_l)/(\mathrm{Im}\, cl) \otimes \mathbb{Z}_l$ is trivial.

We apply Proposition 1.3 which tells as well that over $\mathbb{C}$, the group $Z^{2n-2}(X) \otimes \mathbb{Z}_l$ is locally deformation invariant for families of smooth rationally connected varieties. Note that our smooth projective rationally connected variety X is the fiber X_t of a smooth projective morphism $\phi : \mathcal{X} \to B$ defined over a number field, where $\mathcal{X}$ and B are quasiprojective, geometrically connected and defined over a number field. By local deformation invariance, the vanishing of $Z^{2n-2}(X) \otimes \mathbb{Z}_l$ is equivalent to the vanishing of $Z^{2n-2}(X_{t'}) \otimes \mathbb{Z}_l$ for any point $t' \in B(\mathbb{C})$. Taking for t' a point of B defined over a number field, $X_{t'}$ is defined over a number field. Hence it suffices to prove the vanishing of $Z^{2n-2}(X) \otimes \mathbb{Z}_l$ for X rationally connected defined over a number field L.

We have

$$Z^{2n-2}(X) \otimes \mathbb{Z}_l = H^{2n-2}(X, \mathbb{Z}_l)/(\operatorname{Im} cl) \otimes \mathbb{Z}_l,$$

and by the Artin comparison theorem (cf. [17, Chapter III,§3]), this is equal to

$$\frac{H_{et}^{2n-2}(X, \mathbb{Z}_l(n-1))}{(\operatorname{Im} cl) \otimes \mathbb{Z}_l} = Z_{et}^{2n-2}(X)_l$$

since $H_{et}^{2n-2}(X, \mathbb{Z}_l(n-1))$ consists of Tate classes. Hence it suffices to prove that for X rationally connected defined over a number field and for any l, the group $Z_{et}^{2n-2}(X)_l$ is trivial.

We now apply Proposition 1.5 to X and its reduction X_p for almost every closed point $p \in \operatorname{Spec} \mathcal{O}_L$. It follows that the vanishing of $Z_{et}^{2n-2}(X)_l$ is implied by the vanishing of $Z_{et}^{2n-2}(X_p)_l$. According to Schoen's theorem 3.1, the last vanishing is implied by the Tate conjecture for degree 2 Tate classes on smooth projective surfaces. ■

Remark 3.4. This argument does not say anything on the groups $Z_{rat}^{2n-2}(X)$, since there is no control on the 1-cycles representing given degree $2n-2$ Tate classes on varieties defined over finite fields. Similarly, Theorem 1.1 does not say anything on $Z_{rat}^4(X)$ for X a rationally connected threefold.

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